Exercise 16

Find an equation for the plane that passes through

(a) $(0,0,0)$, $(2,0,-1)$, and $(0,4,-3)$.

(b) $(1,2,0)$, $(0,1,-2)$, and $(4,0,1)$.

(c) $(2,-1,3)$, $(0,0,5)$, and $(5,7,-1)$.

Solution

A particular plane is specified by a normal vector $\mathbf{n}$ and a point with position vector $\mathbf{r}_0$ that lies in it. The equation for this plane comes from the fact that the dot product of $\mathbf{n}$ with any vector in the plane is zero.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Part (a)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0,0,0)$ to $(2,0,-1)$ and $(0,4,-3)$, respectively.

$$\begin{aligned}\mathbf{r}_1 &= (2,0,-1) - (0,0,0) = (2,0,-1)\\ \mathbf{r}_2 &= (0,4,-3) - (0,0,0) = (0,4,-3)\end{aligned}$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix} = (0+4)\hat{\mathbf{x}} - (-6-0)\hat{\mathbf{y}} + (8-0)\hat{\mathbf{z}} = 4\hat{\mathbf{x}} + 6\hat{\mathbf{y}} + 8\hat{\mathbf{z}} = (4,6,8)$$

Therefore, the equation for the plane is

$$(4,6,8) \cdot (x-0, y-0, z-0) = 0$$

$$4(x-0) + 6(y-0) + 8(z-0) = 0$$

$$4x + 6y + 8z = 0$$

$$2x + 3y + 4z = 0.$$

Part (b)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(1,2,0)$ to $(0,1,-2)$ and $(4,0,1)$, respectively.

$$\begin{aligned}\mathbf{r}_1 &= (0,1,-2) - (1,2,0) = (-1,-1,-2)\\ \mathbf{r}_2 &= (4,0,1) - (1,2,0) = (3,-2,1)\end{aligned}$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (-1-4)\hat{\mathbf{x}} - (-1+6)\hat{\mathbf{y}} + (2+3)\hat{\mathbf{z}} = -5\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + 5\hat{\mathbf{z}} = (-5,-5,5)$$

Therefore, the equation for the plane is

$$(-5,-5,5)\cdot(x-1,y-2,z-0)=0$$

$$-5(x-1)-5(y-2)+5(z-0)=0$$

$$-5x+5-5y+10+5z=0$$

$$-5x-5y+5z=-15$$

$$x+y-z=3.$$

Part (c)

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the displacement vectors from $(0,0,5)$ to $(2,-1,3)$ and $(5,7,-1)$, respectively.

$$\mathbf{r}_1=(2,-1,3)-(0,0,5)=(2,-1,-2)$$
$$\mathbf{r}_2=(5,7,-1)-(0,0,5)=(5,7,-6)$$

Take the cross product of these two to obtain a vector normal to the plane that they reside in.

$$\mathbf{n}=\mathbf{r}_1\times\mathbf{r}_2=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 2 & -1 & -2\\ 5 & 7 & -6\end{vmatrix}=(6+14)\hat{\mathbf{x}}-(-12+10)\hat{\mathbf{y}}+(14+5)\hat{\mathbf{z}}=20\hat{\mathbf{x}}+2\hat{\mathbf{y}}+19\hat{\mathbf{z}}=(20,2,19)$$

Therefore, the equation for the plane is

$$(20,2,19)\cdot(x-0,y-0,z-5)=0$$

$$20(x-0)+2(y-0)+19(z-5)=0$$

$$20x+2y+19z-95=0$$

$$20x+2y+19z=95.$$